5 Fractal and Brownian Motion

5.1 Fractal

Mandelbrot (1982) first introduced an idea, called the "fractal", to statistically describe geometrically random shapes, including coast lines, rivers, mountains, clouds, lightning tracks, etc., which are widely observed in nature. According to Mandelbrot, there are many examples including everywhere nondifferentiable figures in nature and he called them fractals. The classification of those figures may be made in terms of invariance and dimensions. More specifically, a conditional requirement under which the invariance holds in fractal is self-similarity. Fractal dimensions can be fractional numbers, which are characteristic of figures under self-similarity (Stauffer and Stanley, 1990; Schroeder, 1991). The idea has been widely extended (see, for instance, Pietronero and Tosatti, 1986).

First, we consider a non-random fractal as an example. In a right triangle, let us draw the altitude above the hypotenuse (see Fig. 5.1). Then, we have three similar right triangles, ΔABC, ΔDBA, and ΔDAC. In Euclidean geometry, because they are similar, the areas of these triangles are proportional to the squares of the corresponding sides with the same proportionality constant, k.

$$\text{Area}[\Delta\text{ABC}] = ka^2\,;\ \text{Area}[\Delta\text{DAC}] = kb^2\,,\ \text{Area}[\text{DBA}] = kc^2\,. \tag{5.1}$$

These relations are invariant with the same value of k. We can repeat the similar procedure on those triangles, to construct an infinite number of similar triangles which satisfy Eq. 5.1 with the same value of k. Thus, these triangles satisfy the condition of a fractal.

Here, it is interesting to note that since area[ΔABC] = area[ΔDAC] +area[ΔDBA], we obtain the famous Pythagorean theorem,

$$a^2 = b^2 + c^2$$

Contrarily to the non-random fractall, Brownian motion is an example of random fractals. Fractals in nature belong to this category. There are many objects, which themselves are not fractals, but some of their properties, after statistical

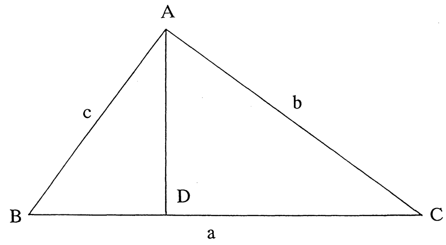


Fig. 5.1 Similar triangles.

averages, may be found to satisfy requirements of fractals. For instance, as shown in Sec. 4.1, the displacement of a Brownian particle satisfies

$$\langle r^2 \rangle = 6Dt \tag{5.2}$$

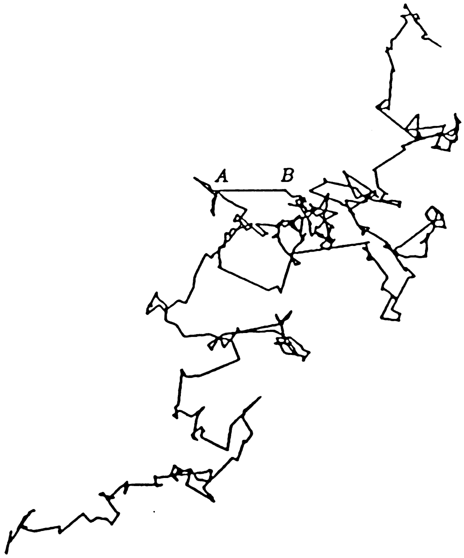
This relation implies that if we measure the particle displacement in a fixed interval of time t and average the squares, the proportionality constant is $6D$ in three dimensions. The value of the proportionality constant does not depend on the value of the time interval, implying that the scaling theorem holds.

Suppose that we photograph the same Brownian particle on the same film many times at a constant time interval, say t. Thus, the positions of the particle are recorded. We join by straight lines the successively photographed positions to obtain Fig. 5.2 (A). It shows a typical pathway in a two-dimensional projection of portraying Brownian motion. Denote the ends of one of the straight-line segments by A and B, corresponding to the time interval t. If we photograph the particle 100 more times during t while the particle is between A and B, we will then see a more detailed behavior of the motion. The result (magnified 10 times) is shown in Fig. 5.2 (B). We can repeat this procedure many times, and then we obtain pictures statistically similar to Fig. 5.2 (B). Thus, we see that Brownian motion is statistically self-similar and we can also find that Eq. 5.2 is invariant when rescaling t. More precisely, geometric figures whose parts can be brought into correspondence with the whole by scaling different directions with different factors are called self-affine. Therefore, Brownian motion is statistically self-affine.

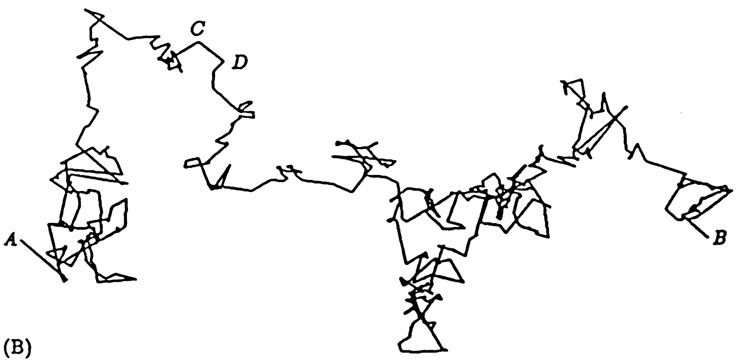
The natural Brownian pathways occur by random collisions. Thus, the pathways are continuous but not differentiable at almost every point on the path. Such a nondifferentiable behavior is universally observed in nature (Barnsley, 1988). Following Mandelbrot, we can treat such nondifferentiable curves by the concept of fractals.

There are abundant examples of fractals, such as Brownian motion, particle diffusion, coast lines, rivers, clouds, colloidal clusters, solid fractures, dielectric breakdown, etc. Thus, fractals may not be expressed in terms of analytical relations, like Eqs. 5.1 or 5.2. The grouping and classification of fractals are then made by self-similarities and numerically by introducing fractal dimensions.

Later (see Sec. 13.2) we will discuss diffusion controlled aggregation in colloidal solutions and we will see how important the idea of fractals is in understanding clusters and gel formation.



(A)



(B)

Fig. 5.2 (A) Brownian pathway. (B) The segment AB of (A), sampled 100 more frequently and magnified 10 times (Schroeder, 1991, with permission from Freemann).

5.2 Fractal and Dimension

One of the classical examples of self-similarity is the Koch curves, proposed in 1904. The shape (total length: L) of Fig. 5.3 A is called the generator. If equilateral triangles are elected over the middle thirds of the straight line segments, we obtain Fig. 5.3 B. The total length of the fractured line is $(4/3)L$. We can repeat

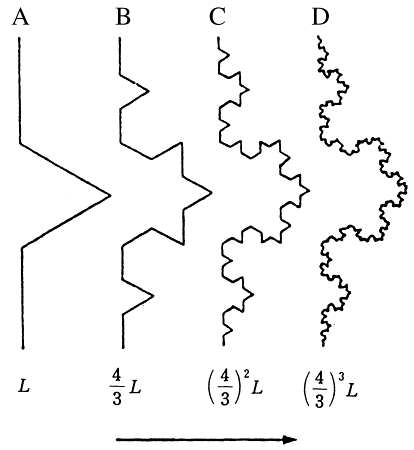


Fig. 5.3 Koch curves.

the similar procedures to obtain Figs. 5.3C and 5.3D, with length $(4/3)^2L$ and $(4/3)^3L$, respectively. The n-th repetition gives the total length of $(4/3)^nL$. Observe that these are self-similar. If the number of straight line segments in the n-th repetition is denoted by N $(=4^{n+1})$, and the length of each straight line segment is by r, $N{\cdot}r$ is the total length, $(4/3)^nL$. Then, the dimension, D, of the figures is defined by the critical value which makes $N{\cdot}r^D$ finite as $n \to \infty$. So it is, if $4/3^D = 1$, since $N{\cdot}(r/L)^D = (4/4^D)(4/3^D)^n$. Therefore, the fractal dimension of the Koch curves is given by

$$D = \ln 4 / \ln 3 = 1.26 \tag{5.3}$$

The definition of the fractal dimension agrees, as shown below, with the traditional dimension of geometry. Equation 5.3 is, in fact, intuitively reasonable. Note that the total length of the Koch curve diverges as $n \to \infty$. An infinitely long curve ending at two points in space is not only just a one-dimensional object, but also does not quite cover a two-dimensional area. The reason that the dimension is different from one is that the Koch curve is subject to the repeated fracturing process and never becomes smooth.

Now, consider, for simplicity, a square (side length: L) on a flat plane. The sides are halved and four similar squares are constructed to fill the original square. This procedure may be repeated many times. At the n-th repetition, the number of N squares to fill the original square is 4^n and the side length, r, is $(1/2)^nL$. Then, we see (Exercise 5.2) that the critical value of D, which makes $N{\cdot}r^D$ finite in the limit of $n \to \infty$, is two, in agreement with the traditional dimension of the figure.

Equivalently, the fractal dimension can be defined as follows. Consider a given smooth area in two dimensions. Suppose that the area is just covered without overlapping, by a number of N small circle of radius r. The number can be shown (Exercise 5.3) to be proportional to r^{-2}. The absolute value of this exponent is the dimension.

If the geometrical figure is in three dimensions, the number of small spheres of radius, r, which just fill the figure, may be counted to find the dimension. Recently, the idea of fractals has been applied to turbulence, diffusion limited aggregation (DLA), strange attractors, etc. and the extension of the definition of fractal dimensions has been introduced (Halsey et al., 1986).

When the fractal dimension is tried to be determined for growing structures in practice, the definition given for the fractal dimension turns out to be ineffective or impractical. There are experimental, computer, and theoretical methods. Some of them will be discussed later, such as in Sec. 6.3, But for more details see Vicsek (1992).

5.3 Brownian Fractal Dimension

Since Eq. 5.2 holds for Brownian motion, the pathways must have dimension of 2. This dimension can also be obtained in the similar way as that of Koch curves, etc. Note that the portion, AB, of Fig. 5.2 A and Fig. 5.2 B correspond to the same Brownian motion between spatial points A and B, but with different resolutions. Fig. 5.2 B has a higher resolution because of more frequent photographing by 100 times. Therefore, Fig. 5.2 B has 100 times more pieces belonging to the same portion of the Brownian motion between A and B. For Brownian motion, we have the relation, $t \propto r^2$, where t is the time interval during which the displacement, r, of the Brownian particle is measured. If the particle is photographed 100 times more during the same time interval, the new time interval is smaller by a factor of 1/100 and the number, N, of measured displacements is larger by a factor of 100. This experiment may be repeated n times by increasing the frequency of photographing by a factor of 100. Then, at the n-th experiment, the time interval is shorter by a factor of $1/100^n$, so that $N=100^n$ and r is smaller by a factor of $1/10^n$. Therefore, the critical value of D, such that $N \cdot r^D$ remains finite as $n \to \infty$, is given by

$$D = \ln N / \ln(1/r) = 2 \tag{5.4}$$

The fractal dimension of Brownian pathways is two, irrespective of the dimension (≥ 2) of space in which the particle moves.

For the one-dimensional space in which the particle is embedded, the Brownian motion cannot have fractal dimension $D=2$. In this case the trajectory (path) passes infinitely often within a given distance of an arbitrary point.

Brownian motion in a space of two dimensions could be plane-filling, but it is not. In fact, there is much self-overlapping as shown in Figs. 5.2 A and B. Thus, if embedded in two dimensions, the probability of returning to the neighborhood of a given location, no matter how narrowly defined, is one. But, in three dimensions the probability of returning the neighborhood is less than one. This conclu-

sion is important to understand why many life-sustaining chemical reactions of, say, enzymes occur on the surface of a cell.

5.4 Power Law (Scaling Invariance)

We have identified Brownian motion as a fractal from the shape of a Brownian trajectory. Since collisions of a Brownian particle with the molecules of the medium are independent, the dimension was obtained from an averaged behavior of individual Brownian particles, Eq. 4.6, which obeys a power law. We can write it in a different way. Consider a random walk. The number N of the steps contained in the walk varies as the square of the end-to-end length r of the walk, i.e., $N \propto r^2$.

There are various power laws we know of. One of the well-known power laws is the gravitational or Coulombic inverse square law (force: $F(r) \propto r^{-2}$). These are invariant under rescaling the distance by multiplying by a constant (scale invariant), or self-similar. They are, therefore, fractals. The dimension of such fractals is found from the relation (Gouyet, 1996):

$$F(\alpha r) = \beta F(r) = \alpha^D F(r)\,, \quad (r \text{ large}) \tag{5.5}$$

with a scaling constant α Therefore, the inverse square law has the fractal dimension of $|D| = 2$. This definition of the dimension agrees with the previously described definition (Exercise 5.5). (However, as shown with Eq. 5.6 or in Sec. 6.3, the interpretation of the dimension is not straight forward.) The Yukawa potential is, on the other hand, proportional to $e^{-\kappa r}/r$ and is not a fractal since it is not self-similar under scaling.

There are power laws whose exponents are not integers. For instant, the acoustic sensation has been known to obey a power law. For a sound to double in loudness L, its intensity I must be multiplies by a factor of 10. This is true over the intensity range of about 12 orders of magnitude beyond which the human ear may perceive with pain. Therefore, we have, noting that $\log_{10} 2 = 0.3$ and

$$L \sim I^{0.3} \tag{5.6}$$

Of course, this relation relies on a statistical average. But, because the logarithm is involved, the error coming from perceptual uncertainties can be small.

An important example of power laws appears in a small angle x-ray scattering or a small angle neutron scattering from disperse systems. (The following discussion is, of course, applicable equally well to light scattering.) We will see in Sec. 6.3 that an x-ray scattering intensity $I(q)$ is given as a function of the magnitude q of the scattering vector (see Eq. 6.10) in terms of a power law (Eq. 6.64):

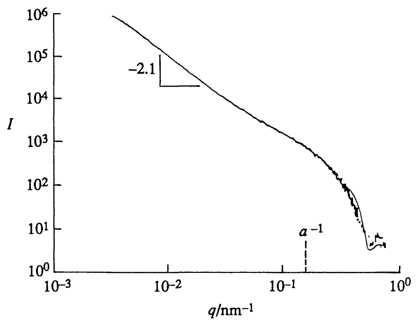


Fig. 5.4 Small angle x-ray scattering from gold aggregate (a: particle size) (Dimon et al., 1986).

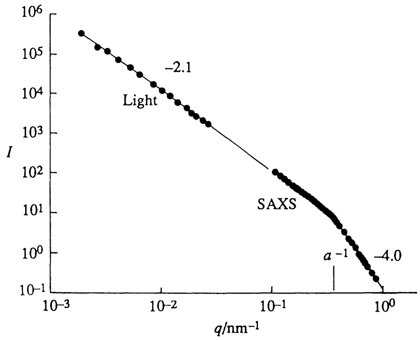


Fig. 5.5 Light and smal angle x-ray scattering (SAXS) of silica aggregate (a: particle size) (Schaefer et al., 1984).

$$I(q) \sim q^{-x} \tag{5.7}$$

within a certain range of q. The value of x depends on the geometrical atructure of a dispersed aggregate of particles in the system (see Figs. 5.4 and 5.5). The distribution of particles in the aggregate may obey a power law (Eqs. 6.64c and 13.2). X-rays are scattered by electrons. Hence, particles of the fractal aggregate are individually an independent scattering unit, but x-rays scattered by them interfere one another. Thus, the particle distribution determines the scattered intensity from the aggregate and, accordingly, the value of x. In Figs. 5.4 and 5.5, aqueous gold and silica dispersions are electrostatically stabilized, but aggregation is induced by reducing the surface charge. The reduction may be done by displacing adsorbed ionic groups from the gold and altering the pH for silica, as well as increasing the ionic strength for the latter. With partial removal of the surface charge, the gold as well as silica aggregates appear to satisfy the following relation between the average number N of particles in an aggregate and the average radius R of the aggregate

$$N = (R/a)^D, \quad (1/R < q < 1/a)$$

with $D \sim 2.1$, where a is the size of a particle. The partial removal of the surface charge corresponds to a slow process of aggregation. If the removal of the surface

charge is complete, the rapid aggregation is promoted with $D \sim 1.8$ (Dimon et al., 1986). For rapid aggregation, cluster-cluster aggregation is understood, while the slow process follows from adding one particle to clusters at a time. Since particles are easily allowed to penetrate into the cluster before sticking, particle-cluster aggregation results in a larger value of the fractal dimension D. If internal rearrangement occur readily, D will approach 3.0. On the other hand, Figs. 5.4 and 5.5 for $q > 1/a$, the value of x (Eq. 5.7) has distinctly larger than 3.0, calling for a different interpretation. It turns out to be related with scattering from the surface structure of aggregates (see Sec. 6.3).

A polymer chain in a good solvent is known to be a fractal as long as the interaction between chains can be neglected. The shape of a polymer chain is important in the steric stabilization of dispersions (see Sec. 11.3). In order to investigate the polymer behavior in a good solvent, neutron scattering (see Sec. 6.4) was carried out on polystyrene dissolved in benzene by Farnoux (1976), showing that Eq. 5.6 holds with $x \sim 5/3$. The value of 5/3 was theoretically predicted by Flory (1971). Each segment of a linear polymer in a good solvent can freely bend like in a random walk. But segments will avoid self-intersection of the chain because of strong repulsions between segments. This repulsion is understood to be due to an effect of the good solvent. Thus, the shape can be described as a self-avoiding random walk (SAW) rather than as a random walk (Brownian motion).

On the other hand, if a linear polymer is dissolved in a bad solvent, the attractive forces between segments are important. Unless the temperature is not high enough, the polymer collapes to a dense structure, behaving as a Euclidean-three-dimensional body.

Exercises

5.1 In Eq. 5.1, the set of triangles must be a fractal. What is the fractal dimension?

5.2 Following the definition of the fractal dimension, described in the text, show in detail that the dimension of a square is two.

5.3 Consider that we try to just fill the area of a given square by a number of N small circles of radius r. Show that $N \propto 1/r^2$.

5.4 Can we consider the two-dimensional fractal (Brownian motion) embedded in the one-dimensional space?

5.5 Show with Koch curves (the total length replacing $F(r)$ of Eq. 5.5) that the definition of the fractal dimension, Eq. 5.5, gives the same value as given by the definition described in Sec. 5.2.

5.6 In the Yukawa potential, $1/\kappa$ is the range ($\sim 10^{-13}$ cm) of the potential between two nucleons. If we propose that this potential is due to an exchange of a particle (called meson) between two nucleons, what do you expect for the mass of the particle? From that uncertainty principle, $\Delta p \cdot \Delta x \sim \hbar$, the range of the particle must be approximately given by $\hbar/mc$, where m and c are the mass of the particle and the speed of light, respectively. Find the value of m. Is the Yukawa potential a fractal in the limit of $m \to 0$?

5.7 An acoustic intensity or an electric power is very often expressed in units of decibel. How is this related to power laws, Eq. 5.5?

5.8 Find the fractal dimension of the self-avoiding random walk in Euclidean three-dimensional space. Is this the same as the exponent of q of $I(q)$ in Eq. 5.6 for polymers in a good solvent?

References

Barnsley, M., “Fractals Everywhere”, Academic Press, San Diego (1988).

Dimon, P., Sinha, S.K., Weitz, D.A., Safinya, C.R., Smith, G.S., Varady, W.A., and Lindsay, H.M., *Phys. Rev. Lett. 57*, 595 (1986).

Farnoux, B., *Thesis*, Université Louis Pasteur, Strasbourg, 1976, cited in Gouyet (1996).

Flory, P.J., “Principles of Polymer Chemistry”, Cornell University Press, Ithaca, N.Y. (1971).

Gouyet, J.F., “Physical and Fractal Structures”, Masson, Springer-Verlag, Paris (1996).

Halsey, T.C., Jensen, M.H., Kadanoff, L.P., Procaccia, I., and Stavans, J., *Phys. Rev. A 33*, 1141 (1986).

Mandelbrot, B., “The Fractal Geometry of Nature”, Freemann, San Francisco (1982).

Pietronero, L. and Tosatti, E., eds., “Fractals in Physics”, North-Holland, Amsterdam, Oxford, New York, Tokyo (1986).

Schaefer, D.W., Martin, J.E., Wiltzius, P., and Cannell, D.S., *Phys. Rev. Lett. 52*, 2371, (1984).

Schroeder, M.R., “Fractals, Chaos, Power Laws”, Freeman, San Francisco (1991).

Stauffer, D. and Stanley, H.E., “From Newton to Mandelbrot”, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo (1990).

Vicsek, T., “Fractal Growth Phenomena”, 2nd ed., World Scientific Publishing Co., Singapore (1992).